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Distance Decay Sensitivity Criterion in Facility Location Problems Under Global Distance Optimization and Facility Attractiveness Strategy.

Festus Owu
Department of Mechatronics Engineering,
Federal University Otuoke, Bayelsa State, Nigeria.
owufu@fuotuoke.edu.ng
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Abstract
Facility location under global distance minimization requires that the optimal values of the facility coordinates be determined such that the entire distance between the facility and its demand points within the service domain is minimized. Since this is an issue where it is assumed that the closer a facility is to its demand points, the higher its patronage, it excludes the possibility of a potential customer overlooking the distance effect and patronizing a faraway facility due to some attractive attributes of such a distance facility. In this study, optimal facility location is done using the gravity method of the minimization of a Euclidean straight line distance function. A distance decay sensitivity parameter is determined through a linear regression modelling of customer's patronage from an assigned customer's population pool, alongside facility attractiveness modelling. It is found that substantial differences arise in possible decisions of an investor with respect to profit maximization, as it relates with the global distance minimization and the attractiveness strategy. For the global distance optimization approach, the entire weighted distance between the facility and the demand points in the numerical validation is reduced from 1,394.0km to 1,220km, a difference of 174km. an investor thus assumes that this distance cut will be compensated for with high patronage. Nevertheless, the determined distance decay factor of 0.1, shows that the decision of any demand point to patronize the located facility, is not significantly distance based. This position is further reaffirmed by the result of the attractiveness analysis, as the shortest weighted distance between one of the demand points (84.853km) was second in the attractiveness rating. (2572), behind (25.756km) of a weighted distance of (273.004km). The demand point with weighted distance of (834.444km) from the facility had a higher attractiveness rating of (2563) than the demand point with a weighted distance of (201.246km) with an attractiveness rating of (1,240).

Keywords: *Facility-Location: Distance-Decay: Attractiveness: Gravity Model: Bonus: Global Distance.*

1. Introduction

Various variables of interest exist for consideration in the quest for deciding the optimal location of a facility in a service domain, which includes, travel times closely related to distances, costs and demand, Ho et al (2008). The objective more often than not, is the task of the determination of the optimal values of any or combination of these variables that will minimize an objective function that is dependent on the related variables, Bleak and Beam (2008), Wilfred et al (2012) and Nahmias (2009). Though the primary aim of facility optimal location is for effective and efficient

service delivery to the customer at an affordable cost, it is also part of the far-reaching desire to meet the service provider's desire of profit maximization for an investor, reflected in high volume of patronage and for equality in assessment and maximum utilization of facilities for measurable improvement in standard of living Zhang et al (2019) and Drezner (2012) and Wilson (1974).
Distance based facility's optimal location models exist in diverse forms in the facility location domain, where the nature of the computed distance could be Euclidean (squared and straight line) Nahmias (2007), Wilfred (2012), (rectilinear or metropolitan) George et al (2001) Dearing and Segars (2022) and Tchebycher, Mahmood et al (2012).
The objective in any of these distance-based optimization function, is to minimize the global distance between the demand points and the located facility, to minimize the maximum distance between any of the point and the sited facility Nahmias (2009).
The emphasis on the distance as the basis for the optimal location of facilities creates the impression that proximity is the major determinant of patronage in a competitive environment or as a motivation for utilization of a facility, as though the consumer is constrained to do so, Drezner (2012), Hoefling (1929). Beyond distance are incentive packages (quantity discount, bonuses on purchased items or on facility utilization etc.), aesthetics, hospitality treat, and other Sunday measures that can become positive influence or attraction that may override distance in influencing patronage, Drezener (2012) in his review of competitive facility location in the plane, considered this issue of influences on patronage other than distance as the concept of attractiveness in competitive business environment with multiple facilities. The concept of competitiveness here looks as though there is a chain of similar facilities with some operations and product from where willing customers can make their choices on which to patronize. In the real world, only on very rear occasions will a facility stand in isolation of others sharing similar products even if at some distance from other, hence conferring the choice decision on potential customers, thereby eliciting competitiveness in spite of spatial considerations. From the fore going, the maximization of the market share of the demand population is essential for the maximization of profit for an investor, Plastria and Carrizosa (2004), and Drezner (2009). The literature on facility location in competitive environment where the market share of each facility is approximately modelled includes, Hofelling (1929), Suruki et al (2007), Hodgson (1978), Re Velle (1986) and Plastria (2001).
Some of the methods of estimating the market share of any facility includes; the proximity, deterministic utility, random utility, cover-based and gravity-based approach, Hotelling (1929), Suzuki et al (2007), Drezner et al., (1998), Drezner et al (2011), Drezner (1995), Aboolian et al (3007) and Bozkaya et al (2010).

The determination of the attractiveness of a facility on a demand point which influences the share of the demand to be satisfied by the facility helps to establish a trade-off in optimal location of a facility that takes into account the influence of distance and the volume of patronage.
In this study, the gravity distance function, involving straight line Euclidean distances will be used to determine the optimal distance location of a single facility with respect to several demand points and a distance sensitivity test will be conducted alongside the determination of the market share value of the facility at the demand points. The optimal location due to the proximity will be compared to that of the share market values for location decision on a site of higher profit for investors and efficient and affordable service for the consumers.

2. Materials and Methods

This section will present the materials required and the methods to be adopted in this study.

2.1 Materials

Published materials on single facility locations servicing various demand points especially on those modelling weighted straight line Euclidean distances between facility and demand points will be required for a framework review. Data for numerical evaluation will be sourced from secondary sources.

2.2 Methodology

In this study, the gravity method will be employed in the determination of the optimal location of a facility from a Euclidean straight line distance function. Subsequently, a linear utility model for the attractiveness of each demand point will be done in conjunction with a gravity-based estimation of a

Distance decay factor, to analysis whether the optimal distance location or the location by market shared value is more justified.

2.3 Model Development

Consider a facility "K" existing in a Cartesian plane "S" with global coordinates "X and Y". Let demand points D be located in the S plane (service domain) with coordinates a_i and bi where I = 1, 2, 3…. n, that ai and bi are the local coordinates of the demand points.

Under the Euclidean straight line distance measures, the global distance function (f (x, y)) to be minimized for the optimal facility (K) location with respect to the demand points is given according to Nahmias (2009) as;

$$f\ (x, y) = \Sigma_{i=1}^{n} \sqrt{(x^{*}\ -\ a_i)^2\ +\ (y\ -\ b_i)^2} \qquad (1)$$

Where W_i the traffic weight attached to each Euclidean distance between the demand points. Depending on the variable of interest existing at the demand point (volume of demand, transportation frequency, utilization rate etc.), the weight can have different form. For the study under consideration, the volume of flow of the population in demand point i into or utilizing facility K will be considered asW_i.

The objective of the distance function of eqn. (1) is to determine the optimal value of X^* and Y^* that will minimize the global distance function reported in Wilfredo et al (2012) and Nahmias (2009);

$$f\ (x^*, y^*) = \min_{x,\, y} f\ (x, y) \qquad (2)$$

Due to the non-convergence or non-closed form of eqn. (1), its solution mathematically, is a lot difficult than it is with the squared Euclidean and rectilinear distances. According to Drezner (1998) and Nahmias (2009), the solution method to equation (1) is an iterative algorithm that requires convergences of values of the facility coordinate from the gravity approximation method. Francis and White (1974), stated that provided the facility location of a new facility (K) does not overlap the location of an existing facility, then a distance function $\emptyset\ (x, y)$ can be defined, which can provide a solution to equation (2). The said distance function $\emptyset\ (x, y)$ according to Nahmias, (2009) is given as;

$$\emptyset_i(x, y) = \frac{w_i}{\sqrt{(x-a_i)^2+(y-b_i)^2}} \qquad (3)$$

Supposing the initial coordinates x_0 *and* y_0 of the facilities K are given, then the new coordinates x_1 *and* y_1 will be given as;

$$x_1 = \frac{\sum_{i=1}^{n} a_i \emptyset_i(x_o, y_o)}{\sum_{i=1}^{n} \emptyset_i(x_o, y_o)} \qquad (4)$$

And

$$y_1 = \frac{\sum_{i=1}^{n} b_i \emptyset_i(x_o, y_o)}{\sum_{i=1}^{n} \emptyset_i(x_0, y_0)} \qquad (5)$$

Given that x_0 *and* y_0 are not known, then they can be evaluated as follows;

$$x_0 = \frac{\sum_{i=1}^{n} w_i a_i}{\sum_{i=1}^{n} w_i} \qquad (6)$$

And

$$y_0 = \frac{\sum_{i=1}^{n} w_i b_i}{\sum_{i=1}^{n} w_i} \qquad (7)$$

In what follows, this starting values of (x_0, y_0) are used to generate other coordinate values until convergences in value is recorded at which point x^* *and* y^* are determined.

Assuming there is a population p_i at demand point i, the proportion (k_i share) of this population ready to access facility (K) according to Wilson, (1976) and Hodgson, (1981) is;

$$\boldsymbol{k_{pi} = e^{\lambda d k_i}} \qquad (8)$$

Where K_{pi} is the share of population i ready to patronize facility K, λ is a distance sensitivity factor that measures the decrease in flow of customer from demand point i to facility K with increase in the distance d_{ki} and d_{ki} is the distance between facility K and the demand point i. The distance d_{ki} is given as;

$$d_{ki} = W_i \sqrt{(x - a_i)^2 + (y - b_i)^2} \qquad (9)$$

The value of the distance sensitivity factor according to Yanguan, (2015) can be evaluated as follows;

$$k_{fi} = \alpha \frac{p_i}{d_{ki}^{\lambda}} \qquad (10)$$

Where k_{fi}, is the flow of customers to facility "K" from demand point(i), $\propto$ is a proportionality constant. The above equation (10) can be transformed into a linear form as;

$$\ln k_{fi} = \ln \propto + \ln p_i - \propto \ln d_{ki} \qquad (11)$$

The above equation (11) in a linear regression form is as follows;

$$Y = \psi + A_i - \lambda X \qquad (12)$$

Where Y = ln k_{fi} (Dependent Variable)

ψ = ln p_i (The estimate of the Y intercept)

A = ln p_i

X = ln d_{ik}

λ = the estimate of the slope of the regression line.

The normal equations of the regression eqn. (12) according to Montgomery et al (2021) and Seber and Lee (2003) are;

$$\psi = \sum y^i - \sum A_i + \lambda \sum X_i \qquad (13)$$

And

$$\lambda \sum X_i^2 = \sum X_i Y_i - \psi \sum X_i - \sum A_i X_i \qquad (14)$$

3. Results and Discussion

The results and the computational details of the numerical evaluation of the analyzed decision-making model is presented below.

3.1 Determination of the Optimal Facility Coordinates.

In the evaluation of the analyzed model, secondary data where sourced from the inflow (patronage) of students from four faculties into the temporary library in a university, which the university authority wishes to relocate to a permanent site in due time, but boarder about where the permanent site should be located for equal access, reduced access encumbrances and retention of highest users of interest. Details of the data collected with respect to average session utilization of the library for six sessions by students of the faculties of Education, Engineering, Health Sciences and Management Sciences, their average faculty population, coordinates, bonuses per each faculty usage and other relevant parameters are given in table 3.1 below.

Table 3.1: Secondary data temporary Library (k), Faculties (Demand Points) coordinates and population and other parameters.

Temporary Library (k)	Faculties i	Library coordinates (X, Y) (m)	Faculties coordinates (a_i, b_i)	Faculties population p_i	Faculties Traffic Weight $(W) = F$	Bonuses B_i
K	EDUCATION		(150,300)	8,000	1,200	2
	ENGINMEERING		(300,600)	9,000	750	7
	HEALTH SCIENCES	(200,250)	(400,350)	4,200	900	5
	MANAGEMENT SCIENCES		(1400,100)	10,000	690	20

From table 3.1, the weighted distances of each faculty (i) to the temporary library (K) can be determined from eqn. (9). Recall eqn. (9)

$$d_{ki} = W_i \sqrt{(x - a_i)^2 + (y - b_i)^2}$$

Hence,

d_{ki} = 84.853km

d_{ki} = 273.004km

d_{ki} = 201.246km

d_{ki} = 834.44km.

Given that $x_0 = 200m$ *and* $y_0 = 250m$, the optimal coordinates X^* *and* Y^* for the permanent library can be determined by the iteration procedure involving equations (3), (4) and (5), which give the optimal value as;

$X^* = 390m$ *and* $Y^* = 354m$

Hence, the minimized global distance, between the permanent Library (L) and the four faculties can be determined from eqn. (2) as follows;

Recall equation (2);

$f(x^*, y^*) = \Sigma_{i=n}^{n} \sqrt{(x^* - a_i)^2 + (y - b_i)^2}$

$f\ (x^*, y^*) = \Sigma_{i=n}^{n} d_{li}$

Where d_{k1} = 295.300km

d_{k2} = 196.460km

d_{k3} = 9.693km

d_{k4} = 718.600km

Hence $f\ (X^*, Y^*) = 1{,}219.955km$

Note that,

$f\ (X, Y) = 1{,}393.547km$

Hence, the minimization process caused a weighted global distance reduction of;

$1{,}393.547 - 1{,}219.955 = 173.592km$

From table 3.1, the library operators place high premium on incentives as bonuses for students accessing the library in form of subsidized transport for long distances. This apparently is to override the hesitancy or drawback of students in further faculties from imbibing reading culture.

From the regression equations of (11) and (12), table 3.2 can be generated. Table 3.2: Regression Equation Parameters

Y_i	X_i	X_i^2	$X_i Y_i$	A_i	$A_i X_i$
7.1	11.4	130.0	80.9	9.0	102.6
6.6	12.5	156.3	82.5	9.1	113.8
6.8	12.2	148.8	83.0	8.3	101.3
6.5	13.5	185.0	88.4	9.2	125.1
2.7	14.7	620.1	334.8	35.6	442.8

Given the entries in table 3.2 and applying the normal equations of (13) and (14), the Y intercept estimate (ψ) and the estimate of the slope of the regression line (λ) which is the distance decay sensitivity factor can be determined.

Recall eqn. (13) and (14).

$\psi = \sum y_i - \sum A_i + \lambda \sum X_i$

And $\lambda \sum X_i^2 = \sum X_i Y_i - \psi \sum X_i - \sum A_i X_i$

$=> \psi = -3.6$

And $\lambda = 0.10$

Hence eqn. (12) can be written as;

$Y = -3.6 + A - 0.1X$

From table 3.1 the operators of the temporary library attach different bonuses to each student from any faculty that patronizes the library resulting in varying attractiveness. Perhaps, the gesture is to encourage students from faculties quite some distances to the library to overcome the drawback associated with costly assess.

To determine the various attractiveness of each faculty to the temporary library, recall eqn. (15).

Recall eqn. (15).

$$A_{ki} = \frac{p_i + B_i}{d_{ki}^{\lambda}}$$

$$A_{ki} = \frac{p_1 + B_1}{d_{k1}^{\lambda}} = \frac{800}{(84{,}853)^{0.1}} = \frac{8002}{3.111} = 2{,}572$$

$$A_{ki} = \frac{p_2 + B_2}{d_{k2}^{\lambda}} = \frac{9000 +}{(273{,}004)^{0.1}} = \frac{9007}{3.496} = 2{,}576$$
$$A_{ki} = \frac{p_3 + B_3}{d_{k3}^{\lambda}} = \frac{4200}{(201{,}246)^{0.1}} = \frac{4205}{3.391} = 1{,}240$$
$$A_{ki} = \frac{p_4 + B_4}{d_{k4}^{\lambda}} = \frac{10000 + 20}{(834{,}444)^{0.1}} = \frac{10{,}020}{3.91} = 2{,}563$$

3.2 Analysis

The results from the optimal location of the facility (permanent library site) based on the minimization of the Euclidean straight line distance's function using the gravity iteration method showed some interesting observations.

First, the faculty of health science with an initial weighted distance of (201.241km) for the temporary library location with a population of 4,200 students, and a patronage of 900 students (population-patronage proportion of about 0.21) was assigned the least optimal weighted distance of (9.693km). Secondly, the permanent library by the optimal strategy, located the faculty further from faculty of education with the highest patronage of 1,200 students of a population of 8,000 students (population-patronage proportion of about 0.15), and closer to the faculty of Health sciences, faculty of Engineering and Management Sciences with combine population-patronage proportion of about 0.36 higher than that of the faculty of Education.

From an investor's perspective reflected in the views of the library operators, locating the library closer to the faculty of science is justified as a marginal increase in student population will result in management patronage higher than any of the faculties hence a shorter distance will resonate with higher patronage. In the second observation, locating the permanent library closer to the faculties of Engineering and Management Sciences with a combine population –patronage proportion of about 0.36, outweighs the gains of locating it at some distance further from the faculty of Education. The optimal distances for the observation allow for customers benefit of reduced assess cost for majority and investors gain drive of higher combined faculties patronage and accessibility for all potential users. In the overall analysis, the distance function minimization reduced the weighted global distance between the faculties and the temporal library from 1,393.547km to 1,219.955km implying shorter distances for higher patronage.

The distance decay sensitivity factor was however determined to be 0.10, suggesting that the decision to patronize the library by student from any faculty was not strongly influenced by distance. What this suggest to a potential investor or the library operator is that, conscious effort must be put in place to reinforce the library with some attribute that constitute higher utility value or attractiveness.

From the perspective of the facility attractiveness strategy in the facility location, the permanent library assuming it remains in the exact position of the temporary library will be more attracted to the faculty of Engineering (A_{k2} = 2,576,) even though it has a lower bonus value than the faculty of Management Science. The faculty of Health Sciences, even though has a shorter weighted distance of (201.246km) to the library than that of the faculty of Management weighted distance of (834.444km) has the least attractiveness to the library.

The result above, explains to the investor that the faculty of engineering has second to the highest students population (9000) and it is at a weighted distance of (273.004km) away from the library compared to the faculty of management science with the highest student population of (10,000) yet at the furthest weighted distance of (834.444km) away from the library hence the faculty of Engineering's higher patronage, hence the need for an investors close attention for profit maximization. The faculty of Health Sciences has the least student's population and it is at a considerable weighted distance (201.246km), third to the longest distance from the facility, hence it is less attractive, implying that an investor even though wishes to increase patronage from increased bonuses, must strongly watch the returns on bonuses from an incremental patronage.

Conclusion

The need for optimal location of a facility for the dual benefit of customer's quality service and minimum assessment cost and the investors profit maximization based on longer share of demand volume cannot be overemphasized. Knowing where to draw the line on proximity criteria and areas to focuses on for higher patronage, irrespective of the distance factor is keen to the survival of any facility. The analyzed model in this study has successfully resolved the above challenge and has added flexibility to the interpretation of distance-based facility location model.

It has particularly also shown that bonuses and incentives may be good motivation for patronage but do not solely constitute the entirety of attractiveness.

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